## Problem 1.

a) $m\ddot{x}_1 = -kx_1 - 2k(x_1 - x_2)$

$= -3kx_1 + 2kx_2$

$\ddot{x}_1 = -\frac{3k}{m}x_1 + \frac{2k}{m}x_2 \quad (1)$

b) $m\ddot{x}_2 = -2k(x_2 - x_1) - kx_2 + F_0\cos(wt)$

$= 2kx_1 - 3kx_2 + F_0\cos(wt)$

$\ddot{x}_2 = \frac{2k}{m}x_1 - \frac{3k}{m}x_2 + \frac{F_0}{m}\cos(wt) \quad (2)$

c) $q_1 \equiv x_2 + x_1$, $q_2 \equiv x_2 - x_1$ (Otherwise we get a nasty - sign in $q_2$)

$(2)+(1)$ : $\ddot{x}_2 + \ddot{x}_1 = -\frac{k}{m}(x_2 + x_1) + \frac{F_0}{m}\cos(wt)$

$\ddot{q}_1 = -\frac{k}{m}q_1 + \frac{F_0}{m}\cos(wt)$

$(2)-(1)$ : $\ddot{x}_2 - \ddot{x}_1 = -\frac{5k}{m}(x_2 - x_1) + \frac{F_0}{m}\cos(wt)$

$\ddot{q}_2 = -\frac{5k}{m}q_1 + \frac{F_0}{m}\cos(wt)$

$w_1 = \sqrt{k/m}$, $w_2 = \sqrt{5k/m}$

d) $q_1(t) = C_1(w)\cos(wt)$ ~> Steady state Solution

$C_1(w) = \frac{F_0/m}{(w_1^2 - w^2)} \quad w_1^2 = k/m$

$q_2(t) = C_2(w)\cos(wt)$ ~> Steady state Solution

$C_2(w) = \frac{F_0/m}{(w_2^2 - w^2)} \quad w_2^2 = 5k/m$

$q_1 = x_2 + x_1$, $q_2 = x_2 - x_1$

$x_1 = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}\left(\frac{F_0/m}{(w_1^2 - w^2)} - \frac{F_0/m}{(w_2^2 - w^2)}\right)\cos(wt)$

$x_2 = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}\left(\frac{F_0/m}{(w_1^2 - w^2)} + \frac{F_0/m}{(w_2^2 - w^2)}\right)\cos(wt)$

e) $w \longrightarrow w_1 \quad |C_1| >> |C_2|$ ~> $x_1 \approx x_2$

$w \longrightarrow w_2 \quad |C_1| << |C_2|$ ~> $x_1 \approx -x_2$

Problem 2.

a) $y(x,t) = Ae^{-(x-vt)^2/2\sigma^2}$

$$\frac{\partial y}{\partial t} = \frac{(2v)(x-vt)}{2\sigma^2} Ae^{-(x-vt)^2/2\sigma^2}$$

$$= \frac{v}{\sigma^2}(x-vt)\, Ae^{-(x-vt)^2/2\sigma^2}$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{v^2}{\sigma^2} Ae^{-(x-vt)^2/2\sigma^2} + \frac{v^2}{\sigma^4}(x-vt)^2 Ae^{-(x-vt)^2/2\sigma^2}$$

$$= \frac{Av^2}{\sigma^2}\left(\frac{(x-vt)^2}{\sigma^2} - 1\right) e^{-(x-vt)^2/2\sigma^2}$$

$$\frac{\partial y}{\partial x} = -\frac{2(x-vt)}{2\sigma^2} Ae^{-(x-vt)^2/2\sigma^2} = -\frac{(x-vt)}{\sigma^2} Ae^{-(x-vt)^2/2\sigma^2}$$

$$\frac{\partial^2 y}{\partial x^2} = -\frac{1}{\sigma^2} Ae^{-(x-vt)^2/2\sigma^2} + \frac{(x-vt)^2}{\sigma^4} Ae^{-(x-vt)^2/2\sigma^2}$$

$$= \frac{A}{\sigma^2}\left(\frac{(x-vt)^2}{\sigma^2} - 1\right) e^{-(x-vt)^2/2\sigma^2}$$

$$v^2\frac{\partial^2 y}{\partial x^2} = \frac{Av^2}{\sigma^2}\left(\frac{(x-vt)^2}{\sigma^2} - 1\right) e^{-(x-vt)^2/2\sigma^2}$$

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} = \frac{Av^2}{\sigma^2}\left(\frac{(x-vt)^2}{\sigma^2} - 1\right) e^{-(x-vt)^2/2\sigma^2}$$

$\leadsto$ wave equation satisfied

b) $$E = \frac{1}{2}\mu\int_{-\infty}^{+\infty} dx\left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]_{t=t_0}$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \frac{v}{\sigma^2}x\, Ae^{-x^2/2\sigma^2}$$

$$\left(\left.\frac{\partial y}{\partial t}\right|_{t=0}\right)^2 = \frac{v^2}{\sigma^4}x^2A^2e^{-x^2/\sigma^2}$$

$$\left.\frac{\partial y}{\partial x}\right|_{t=0} = -\frac{x}{\sigma^2}Ae^{-x^2/2\sigma^2}$$

$$v^2\left(\left.\frac{\partial y}{\partial x}\right|_{t=0}\right) = \frac{v^2}{\sigma^4}x^2A^2e^{-x^2/\sigma^2}$$

$$\left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]_{t=0} = \frac{2v^2}{\sigma^4}x^2A^2e^{-x^2/\sigma^2}$$

$$E = \mu\left(\frac{Av}{\sigma^2}\right)^2\int_{-\infty}^{\infty}dx\; x^2e^{-x^2/\sigma^2}$$

Since $\int_{-\infty}^{\infty}dx\; x^2e^{-x^2} = \frac{\sqrt{\pi}}{2}$

$$\int_{-\infty}^{\infty}dx\; x^2e^{-x^2/\sigma^2} = \sigma^3\int_{-\infty}^{\infty}du\; u^2e^{-u^2} = \frac{\sigma^3\sqrt{\pi}}{2}$$

$u = x/\sigma$

$dx = \sigma\, du$

$$E = \mu\left(\frac{Av}{\sigma^2}\right)^2\frac{\sigma^3\sqrt{\pi}}{2} = \frac{\mu\sqrt{\pi}(Av)^2}{2\sigma}$$

c) transverse velocity $\frac{\partial y}{\partial t}$ maximized when transverse acceleration $\frac{\partial^2 y}{\partial t^2} = 0$

$$\frac{\partial^2 y}{\partial t^2} = \frac{Av^2}{\sigma^2}\left(\frac{(x-vt)^2}{\sigma^2} - 1\right)e^{-(x-vt)^2/2\sigma^2} = 0$$

$\rightsquigarrow (x-vt)^2 = \sigma^2$

$\rightsquigarrow (x-vt) = \pm\sigma$ (FWHM points)

$$\frac{\partial y}{\partial t} = \frac{v}{\sigma^2}(x-vt)Ae^{-(x-vt)^2/2\sigma^2}$$

$$\left.\frac{\partial y}{\partial t}\right|_{(x-vt)=\sigma} = \frac{v}{\sigma}Ae^{-\sigma^2/2\sigma^2} = \frac{Av}{\sigma}e^{-1/2}$$

$$\left(\left.\frac{\partial y}{\partial t}\right|_{(x-vt)=-\sigma} = \frac{-Av}{\sigma}e^{-1/2}\right)$$

$\rightsquigarrow$ Maximal transverse velocity. $\frac{Av}{\sigma}e^{-1/2}$

Problem 3.

a) $0<x<L/2$ : $f(x) = \frac{2d}{L}x$

$L/2<x<L$ : $f(x) = 2d - \frac{2d}{L}x$

b) $A_n = \frac{2}{L}\int_0^L dx\; f(x)\sin\left(\frac{n\pi x}{L}\right)$

$= \frac{2}{L}\left[\int_0^{L/2} dx\; \frac{2dx}{L}\sin\left(\frac{n\pi x}{L}\right) + \int_{L/2}^{L} dx\left(2d - \frac{2d}{L}x\right)\sin\left(\frac{n\pi x}{L}\right)\right]$

$\frac{2d}{L}\int_0^{L/2} dx\; x\sin\left(\frac{n\pi x}{L}\right) = \frac{2d}{L}\left(-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2} dx\cos\left(\frac{n\pi x}{L}\right)\right)$

$u = x \quad dV = \sin\left(\frac{n\pi x}{L}\right)dx$

$du = dx \quad V = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$\frac{-L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)$ $\qquad$ $\left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} = \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)$

$= \frac{2d}{L}\left(\frac{-L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right)$

$-\frac{2d}{L}\int_{L/2}^{L} dx\; x\sin\left(\frac{n\pi x}{L}\right) = -\frac{2d}{L}\left(-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} + \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L}\right)$

$= -\frac{2d}{L}\left(\frac{-L^2}{n\pi}\cos(n\pi) + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right)$

$2d\int_{L/2}^{L} dx\;\sin\left(\frac{n\pi x}{L}\right) = \frac{-2dL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} = -\frac{2dL}{n\pi}\cos(n\pi) + \frac{2dL}{n\pi}\cos\left(\frac{n\pi}{2}\right)$

$\rightarrow$ putting it all together: $A_n = \frac{2}{L}\left(\frac{4d}{L}\left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right)$

$= \frac{8d}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$

c) $A_n = \frac{8d}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$

$\sin\left(\frac{n\pi}{2}\right) = 0$ for even values of $n$

$n = 2, 4, 6, \cdots 2k$ modes are not excited

d) $E_n = \frac{1}{4}\mu L A_n^2 \omega_n^2$

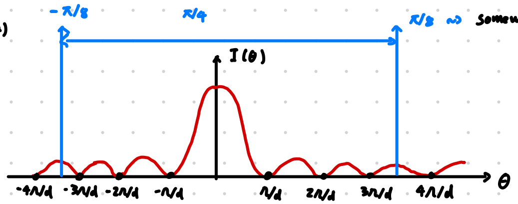
$A_n = \frac{8d}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$

$\omega_n = v\left(\frac{n\pi}{L}\right) = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$

$E_n = \frac{1}{4}\mu L\,\frac{64d^2}{(n\pi)^4}\sin^2\left(\frac{n\pi}{2}\right)\frac{(n\pi)^2}{L^2}\,\frac{T}{\mu} = \frac{16d^2T}{(n\pi)^2}\sin^2\left(\frac{n\pi}{2}\right)$

Problem 4.

a)




⇝ $\theta_n = \frac{n\lambda}{d}$

⇝ by symmetry $\frac{\pi}{8}$ is between $\theta_3$ & $\theta_4$

$$\frac{3\lambda}{d} \leq \frac{\pi}{8} < \frac{4\lambda}{d}$$

⇝ $\frac{3\lambda}{d} \leq \frac{\pi}{8} \longrightarrow \lambda \leq \frac{\pi d}{24}$

⇝ $\frac{4\lambda}{d} > \frac{\pi}{8} \longrightarrow \lambda > \frac{\pi d}{32}$

$$\Bigg\} \quad \frac{\pi d}{32} < \lambda \leq \frac{\pi d}{24}$$

b) in this case $\frac{\pi}{8} = \theta_3 = \frac{3\lambda}{d} \longrightarrow \lambda = \frac{\pi d}{24}$

in new wave length $\lambda'$, $\frac{\pi}{8} = \theta_4 = \frac{4\lambda'}{d} \longrightarrow \lambda' = \frac{\pi d}{32}$

$$\frac{\lambda'}{\lambda} = \frac{\frac{\pi d}{32}}{\frac{\pi d}{24}} = \frac{3}{4} = \frac{\nu}{\nu'}$$

$$\nu' = \frac{4}{3}\nu$$

c) $\sin\theta_1 = \frac{\lambda}{d}$ , $\sin\theta_n = \frac{n\lambda}{d}$

⇝ $\sin\theta_1' = \frac{\lambda}{(d/2)} = \frac{2\lambda}{d}$ , $\sin\theta_n' = \frac{n\lambda}{(d/2)} = \frac{2n\lambda}{d}$

⇝ $\theta$ of the first diffraction minima significantly increases, producing a broader central maximum.

⇝ fewer minima observed on screen